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1979 J. Phys. A: Math. Gen. 12 L163

(http://iopscience.iop.org/0305-4470/12/7/003)

J. Phys. A: Math. Gen., Vol. 12, No. 7, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Non-linear differential-difference equations with N-dependent coefficients II†

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Received 20 February 1979

Abstract. Introducing a new Wronskian relation we enlarge the class of soluble non-linear differential-difference equations related to the discrete Schrödinger spectral problem to include equations with n-dependent coefficients. Within this enlarged class we find a generalised Volterra equation.

It is well known that one can extend the spectral transform to solve non-linear evolution equations with variable (x dependent) coefficients (Newell 1978, Calogero and Degasperis 1978).

We (Levi and Ragnisco 1978, 1979) have been able to treat, by an analogous extension, non-linear differential-difference. Equations (NDDE) with variable coefficients associated with the Zakharov–Shabat spectral problem.

It seems worthwhile to carry out the same extension for the discrete Schrödinger problem, introduced by Flaschka (1974) to solve the Cauchy problem for the Toda lattice, (Toda 1976) and for which Dodd (1978) introduced the Bäcklund transformation. To do this we introduce a new Wronskian relation.

The discrete Schrödinger spectral problem reads

$$a(n-1)\psi(n-1)+a(n)\psi(n+1)+b(n)\psi(n)=\lambda\psi(n) \tag{1}$$

where the eigenvalue $\lambda=(z+z^{-1})/2$ and $a(n,t)$ and $b(n,t)$ are finite-valued time-dependent potentials satisfying the boundary conditions

$$\lim_{|n|\to\infty} a(n)=\tfrac{1}{2}, \qquad \lim_{|n|\to\infty} b(n)=0, \tag{2}$$

with $a(n)$ positive for any integer n.

We can define the spectral data (Flaschka 1974)

$$S\colon \{z_j(|z_j|<1),\, C_j(j=1,\ldots,N);\, R(z)=\beta(z)/\alpha(z)(|z|=1)\} \tag{3}$$

through the asymptotic behaviour, for the N bound states

$$\psi(n,z_j) \underset{n\to+\infty}{\sim} c_j(z_j)^n \tag{4a}$$

† Supported in part by CNR grant N.78.00919.02.

and for the scattering states

$$\psi(n, z) \underset{n\to+\infty}{\sim} \beta(z)z^n + \alpha(z)z^{-n}, \tag{4b}$$

$$\psi(n, z) \underset{n\to-\infty}{\sim} z^{-n} \tag{4c}$$

where $\psi(n, z_j)$ are the normalised bound-state solutions with normalisation constants c_j, $R(z)$ is the reflection coefficient and $T(z) = [\alpha(z)]^{-1}$ the transmission coefficient. For the potentials $a(n, t)$ and $b(n, t)$ we obtain the spectral data S through the solution of the direct problem (1), taking into account the asymptotic properties of the eigenfunctions (4). Starting from the spectral data S, we recover the potentials uniquely by solving the discrete 'integral' equation (Flaschka 1974)

$$K(n, m) + F(n+m) + \sum_{n'=n+1}^{\infty} K(n, n')F(n'+m) = 0 \tag{5a}$$

with the boundary condition

$$(K(n, n))^{-2} = 1 + F(2n) + \sum_{n'=n+1}^{\infty} K(n, n')F(n'+n), \tag{5b}$$

where

$$F(n) = (2\pi\mathrm{i})^{-1} \oint \mathrm{d}z R(z) z^{n-1} + \sum_{j=1}^{N} c_j^2 z_j^n, \tag{5c}$$

which yields the 'potentials'

$$a(n) = \tfrac{1}{2}K(n+1, n+1)/K(n, n), \qquad b(n) = -\tfrac{1}{2}(K(n-1, n) - K(n, n+1)). \tag{6}$$

The associated class of NDDE for $a(n, t)$ and $b(n, t)$ was obtained by Flaschka and McLaughlin (1976) and Dodd (1978) through the construction of the spectral problem for the 'squared eigenfunctions' of (1) (Ablowitz *et al* 1974, Kaup 1976) written for two pairs of potentials $(a(n), b(n))$, $(a'(n), b'(n))$ having the same asymptotic properties(2):

$$\Lambda\begin{pmatrix} v(n) \\ w(n) \end{pmatrix} = \lambda \begin{pmatrix} v(n) \\ w(n) \end{pmatrix}, \tag{7a}$$

$$\Lambda\begin{pmatrix} v(n) \\ w(n) \end{pmatrix} = \begin{pmatrix} \Sigma_{j=-\infty}^{n-1} \{\pi(n, j)[(b'(j) - b(j))v(j) + (a'^2(j) - a^2(j))a'^{-1}(j)w(j)]\} \\ \quad + b'(n)v(n) + a(n-1)w(n-1) + a'(n)w(n) \\ a'^{-1}(n)\, \Sigma_{j=-\infty}^{n-1} \pi^{-1}(n, j)[(a^2(j-1) - a'^2(j))v(j) + a'(j)(b(j) \\ \quad - b'(j+1))w(j)] + a^2(n-1)a'^{-1}(n)v(n) + a(n)v(n+1) \\ \quad + b(n)w(n)), \end{pmatrix} \tag{7b}$$

$$\begin{matrix} v(n) = \psi(n)\psi'(n) \\ w(n) = \psi(n)\psi'(n+1) \end{matrix}, \qquad \pi(n, l) = \prod_{j=l}^{n-1} a'(j)/a(j). \tag{7c}$$

In the limit $n \to +\infty$, $a'(j) = a(j) + (a(j))_t\,\mathrm{d}t$, $b'(j) = b(j) + (b(j))_t\,\mathrm{d}t$ we obtain the following generalised Wronskian-type formulae

$$\alpha^2 R_t = -2/(z - z^{-1}) \sum_{j=-\infty}^{+\infty} [(b(j))_t v(j) + 2(a(j))_t w(j)], \tag{8a}$$

$$\alpha^2 R(z-z^{-1}) = 8/(z-z^{-1}) \sum_{j=-\infty}^{+\infty} \{[a^2(j-1)-a^2(j)]v(j)+a(j)(b(j)-b(j+1))w(j)\}. \tag{8b}$$

A new class of soluble NDDE with coefficients which depend linearly on n can be derived starting from the *new* Wronskian relation

$$\begin{aligned} a(n)\{2(n+2)a(n+1)\psi(n)\psi(n+2)-2na(n)\psi^2(n+1)-(\lambda-b(n+1))\psi(n)\psi(n+1) \\ +(1-\lambda^2)(z-z^{-1})/2z[\psi(n)\psi_z(n+1)-\psi(n+1)\psi_z(n)]\}\big|_{M-1}^{N} \\ = \sum_{n=M}^{N} \{[1-b^2(n)+2(n-1)a^2(n-1)-2(n+1)a^2(n)]v(n) \\ +a(n)[(2n-1)b(n)-(2n+3)b(n+1)]w(n)\} \end{aligned} \tag{9}$$

which holds in the limit $(a'(n), b'(n)) = (a(n), b(n))$. Letting $N\to+\infty$, $M\to-\infty$ and taking into account the asymptotic behaviour of the potentials (2) and of the eigenfunctions (4), we obtain

$$\begin{aligned} \alpha^2 zR_z(z-z^{-1}) = -4/(z-z^{-1}) \sum_{j=-\infty}^{+\infty} \{[1-b^2(j)+2(j-1)a^2(j-1)-2(j+1)a^2(j)]v(j) \\ +a(j)[(2j-1)b(j)-(2j+3)b(j+1)]w(j)\}. \end{aligned} \tag{10}$$

Thus if the following linear partial differential equation is satisfied by the scattering data (3)

$$R_t+(z-z^{-1})(\Omega(\lambda)R+\Omega'(\lambda)zR_z), \tag{11}$$

Ω, Ω' being arbitrary entire functions of the complex variable λ, equations (8) and (10) imply that the following equation must be satisfied:

$$\begin{aligned} \sum_{j=-\infty}^{+\infty} \{(b(j))_t-4\Omega(\lambda)(a^2(j-1)-a^2(j)) \\ +2\Omega'(\lambda)[1-b^2(j)+2(j-1)a^2(j-1)-2(j+1)a^2(j)]\}v(j) \\ +\sum_{j=-\infty}^{+\infty} 2\{(a(j))_t-2\Omega(\lambda)a(j)[b(j)-b(j+1)] \\ +\Omega'(\lambda)a(j)[(2j-1)b(j)-(2j+3)b(j+1)]\}w(j)=0. \end{aligned} \tag{12}$$

Introducing the adjoint L^+ of the operator Λ in the limit $(a'(n), b'(n)) = (a(n), b(n))$† and exploiting the (assumed) completeness of the set of squared eigenfunctions $\binom{v(n)}{w(n)}$, we have the following couple of NDDE:

$$\begin{aligned} \begin{pmatrix} b(n) \\ a(n) \end{pmatrix}_t = \Omega(L^+)\begin{pmatrix} 4(a^2(j-1)-a^2(n)) \\ 2a(n)(b(n)-b(n+1)) \end{pmatrix} \\ -\Omega'(L^+)\begin{pmatrix} 2[1-b^2(n)+2(n-1)a^2(n-1)-2(n+1)a^2(n)] \\ a(n)[(2n-1)b(n)-(2n+3)b(n+1)] \end{pmatrix} \end{aligned} \tag{13}$$

† We notice that the expression of the operator Λ^+ given by Dodd (1978) is not correct.

with

$$L^{+}\begin{pmatrix} p(n) \\ q(n) \end{pmatrix} = \begin{pmatrix} b(n)p(n) + a^{-1}(n)a^{2}(n-1)q(n) + a(n-1)q(n-1) \\ \quad + (a^{2}(n-1) - a^{2}(n)] \sum_{j=n+1}^{\infty} q(j)a^{-1}(j) \\ a(n)[p(n+1) + p(n)) + b(n)q(n) \\ \quad + a(n)(b(n) - b(n+1)) \sum_{j=n+1}^{\infty} q(j)a^{-1}(j) \end{pmatrix}. \tag{14}$$

The solution of eq. (11) via the method of characteristics yields

$$R(z,t) = R(z_0(z,t),0)\exp\left[-\int_0^t \mathrm{d}t'(\zeta - \zeta^{-1})\Omega(\zeta + \zeta^{-1})\right], \tag{15}$$

the function $\zeta(z_0(z,t),t)$ being defined by the differential equation

$$\zeta_b(z_0,t) = \zeta(\zeta - \zeta^{-1})\Omega'(\zeta + \zeta^{-1}) \tag{16a}$$

together with the boundary conditions

$$\zeta(z_0,0) = z_0, \qquad \zeta(z_0,t) = z. \tag{16b}$$

Analogously, for the time evolution of the bound-state parameters we obtain

$$z_{j,t} = z_j(z_j - z_j^{-1})\Omega'(z_j + z_j^{-1}), \tag{17}$$

$$c_j^2(t) = c_j^2(0)\zeta_{z_0}(z_j(0),t)\exp\left[-\int_0^t \mathrm{d}t'(z_j - z_j^{-1})\Omega(z_j + z_j^{-1})\right] \tag{18}$$

where $\zeta_{z_0}(z_j(0),t)$ is the partial derivative with respect to z_0 of the function $\zeta(z_0,t)$, defined by (16*a*, *b*), evaluated for $z_0 = z_j(0)$.

Equation (13) admits an interesting subcase $b(n) = 0$, which is consistent if $\Omega(L^{+}) = \omega(\mathrm{O})L^{+}$, $\Omega'(L^{+}) = \omega'(\mathrm{O})L^{+}$, where $\mathrm{O} = (L^{+})^2$.

In this case equation (13) becomes

$$(a(n))_t = 4\omega(\mathrm{O})a(n)(a^2(n-1) - a^2(n+1)) - 4\omega'(\mathrm{O})a(n)[1 - a^2(n) + (n-1)a^2(n-1) - (n+2)a^2(n+1)] \tag{19}$$

with

$$\mathrm{O}q(n) = (a^2(n) + a^2(n-1))q(n) + a(n)\left\{a(n-1)q(n-1) + a(n+1)q(n+1) + [a^2(n-1) - a^2(n+1)]\sum_{j=n+1}^{\infty} a^{-1}(j)q(j)\right\}. \tag{20}$$

For $\omega = p/8 =$ constant and $\omega' = q/8 =$ constant, introducing the new variable $N(n) = a^2(n)$ we obtain

$$(N(n))_t = N(n)\{(N(n)-1)q + [(n+2)q - p]N(n+1) - [(n-1)q - p]N(n-1)\} \tag{21}$$

which is a generalised Volterra equation with a self interaction term.

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